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Averaged Lagrangians and the Mean Dynamical Effects of Fluctuations in Continuum Mechanics

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Decomposition of multiscale problems & scale-up
In turbulence, in climate modeling and in other multiscale fluids problems, a major challenge is “scale-up.” This is the challenge of deriving models that correctly capture the mean, or large scale flow—including the influence on it of the rapid, or small scale dynamics.

In classical mechanics this sort of problem has been approached by choosing a proper “slow + fast” decomposition and deriving evolution equations for the slow mean quantities by using, say, the method of averages. For nondissipative systems in classical mechanics that arise from Hamilton’s variational principle, the method of averages may extend to the averaged Lagrangian method, under certain conditions.

Eulerian vs Lagrangian means
In meteorology and oceanography, the averaging approach has a venerable history and many facets. Often this averaging is applied in the geosciences in combination with additional approximations involving force balances (for example, geostrophic and hydrostatic balances). It is also sometimes discussed as an initialization procedure that seeks a nearby invariant “slow manifold.” Moreover, in meteorology and oceanography, the averaging may be performed in either the Eulerian, or the Lagrangian description. The relation between averaged quantities in the Eulerian and Lagrangian descriptions is one of the classical problems of fluid dynamics.

This is quite a bonus for both approaches to modeling fluids.

Generalized Lagrangian mean (GLM)
The GLM equations of Andrews & McIntyre (*J. Fluid Mech.* **89** [1978] 609–646) systematize the approach to Lagrangian fluid modeling by introducing a slow + fast decomposition of the Lagrangian particle trajectory in general form. They then relate the Lagrangian mean of a fluid quantity at the mean particle position to its Eulerian mean, evaluated at the displaced fluctuating position. The GLM equations are expressed directly in the Eulerian representation. The Lagrangian mean has the advantage of preserving the fundamental transport structure of fluid dynamics. For example, the Lagrangian mean commutes with the scalar advection operator and it preserves the Kelvin circulation property of the fluid motion equation.

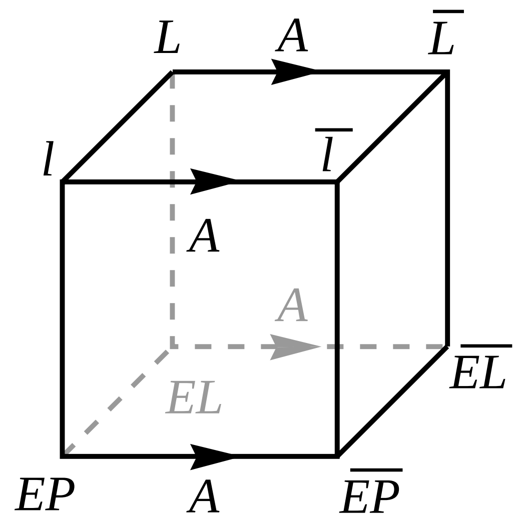

Figure: The Averaged EP theorem produces a cube consisting of four equivalence relations on each of its left and right faces, and four commuting diagrams (one on each of its four remaining faces).

Compatibility of averaging and reduction of Lagrangians for mechanics on Lie groups
In making slow + fast decompositions and constructing averaged Lagrangians for fluid dynamics, care must generally be taken to see that the averaging and reduction procedures do not interfere with each other. Reduction in this context refers to symmetry reduction of the action principle by the subgroup of the diffeomorphisms that takes the Lagrangian representation to the Eulerian representa-

tion of the flow field. The theory for this yields the Euler-Poincaré (EP) equations.

Euler-Poincaré (EP) equations

This compatibility requirement is handled automatically in the GLM approach. The Lagrangian mean of the action principle for fluids does not interfere with its reduction to the Eulerian representation, since the averaging process is performed at *fixed Lagrangian coordinate.* Thus, the process of taking the Lagrangian mean is compatible with reduction by the particle-relabeling group.

We performed this reduction of the action principle and thereby placed the GLM equations into the EP framework. In doing this, we demonstrated the *variational reduction property of the Lagrangian mean.* This is encapsulated in the **Averaged Euler-Poincaré Theorem.***The GLM averaging process preserves the variational structure in the EP framework.*

According to this theorem, the Lagrangian mean's preservation of the fundamental transport structure of fluid dynamics also extends to preserving the EP variational structure of these equations. This preservation of structure is illustrated in the figure. The back face of the cube in the figure displays the preservation of variational structure in the Lagrangian fluid picture. Hamilton's principle with L yields the Euler-Lagrange equations EL in this picture, and GLM averaging at fixed Lagrangian coordinate A preserves this relation. Namely, Hamilton's principle with the averaged Lagrangian L yields the averaged Euler-Lagrange equations EL.

This pair of Hamilton's principles and Euler-Lagrange equations has its counterpart in the Eulerian picture of fluid dynamics—on the front face of the cube—whose variational relations are *also* exactly preserved by the GLM averaging process. The bottom front edge of the cube represents the GLM averaged equations of Andrews & McIntyre [1978]. The six faces of the EP averaging cube in the figure represent six interlocking commutative diagrams that enable modeling and GLM averaging to be performed equivalently either at the level of the equations, as in Andrews & McIntyre [1978], or at the level of Hamilton's principle. At the level of Hamilton's principle, powerful theorems from other mean field theories are available. An example is Noether's theorem, which relates symmetries of Hamilton's principle to conservation laws of the equations of motion. Thus, the GLM averaged Hamilton's principle yields the GLM averaged fluid equations in either fluid picture and one may transform interchangeably along the edges of the cube in search of physical insight.

Of course, this extension and these commuting relationships are not possible with the Eulerian mean, because the Eulerian mean does *not* preserve the transport structure of fluid mechanics.

Thus, the Averaged EP Theorem puts the GLM averaged-Lagrangian approach and the method of GLM-averaged equations onto equal footing. This is quite a bonus for *both* approaches to modeling fluids. The averaged-Lagrangian theory produces dynamics that can be verified directly by averaging the original equations, and the GLM-averaged equations inherit the conservation laws that are available from the symmetries of the Lagrangian.

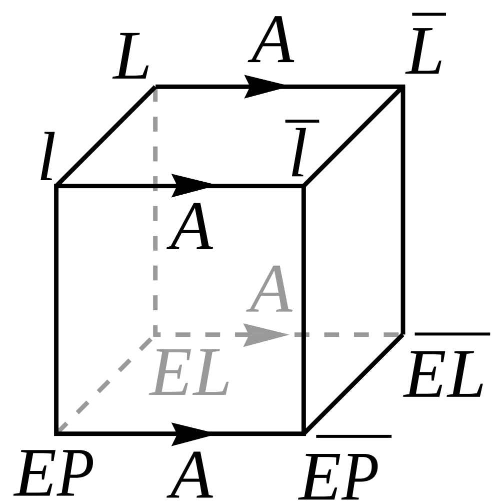

Figure: The Averaged EP theorem produces a cube consisting of four equivalence relations on each of its left and right faces, and four commuting diagrams (one on each of its four remaining faces).

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